Indian Statistical Institute
Second Semester Examination 2003-2004
M.Math I Year
Differential Geometry

Time: 3 hrs Date:03-05-04 Max. Marks : 50

Answer all six questions:

1. Consider the two-dimensional Riemannian manifold (M, g) where $M = \{(x, y) \in I\!R^2 : y > 0\}$ and $g_{(x,y)} = \frac{1}{y^2} dx \bigotimes dx + \frac{1}{y^2} dy \bigotimes dy$

   a) For any $a \in I\!R$, prove that the map $f_a : M \to M$ given by $f_a(x, y) = (x + a,\ y)$ is an isometry.

   b) For any $a \in I\!R$, prove that the curve $\sigma_a : I\!R \to M$ given by $\sigma(t) = (a,\ e^t)$ is a geodesic.

   c) Calculate the curvature of $(M,\ g)$ at any point $(x_0,\ y_0)$. [11]

2. Let M and N be n-manifolds with M compact and N converted. Let $f : M \to N$ be an immersion. Prove that f is onto (surjective). [6]

3. Let $({}_{\pi}\tilde{M},\ \tilde{S}) \downarrow (M,\ g)$ be a Riemannian Covering i.e., a smooth covering such that π is orientation preserving. If the covering is a finite K-sheeted covering prove that Vol $(\tilde{M}, \tilde{g}) = K$ Vol (M, g). [8]
   Hint: Recall the proof when π is actually an isometry.

4. Let M and N be compact oriented n- manifolds. Let Ω be an orientation form for N and let $f,\ g : M \to N$ be two smooth maps. If f and g are smoothly, homotopic, (i.e. if there is a smooth map $F : M \times [0, 1] \to N$ with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all x). Then prove that $\int\limits_M f^*(\Omega) = \int\limits_M g^*(\Omega)$

5. Let (M, g) be a Riemannian manifold and "P" a point in M, Z, a submanifold of M. Let $C : [0, L] \to M$ be a geodesic such that $l(c) = \inf\limits_{x \in A} d(P, x)$

   a) Let C_t be a variation of C and Y be the corresponding variation vector field. We know that $\inf -\epsilon < t\epsilon l(C_t) = l(C)$ for all variations with $C_t(L) \in A$ and $C_t(0) = P$. For variations of this type, what are the restrictions on $Y(t)$ and $Y(0)$?

   b) The first variation formula for the length functional is

$$\frac{d}{dt} l(C_t)\bigg|_{t=0} = \quad \langle y(s),\ C'(s)\rangle \int_0^2 - \int_0^2 \langle y(s), \nabla_C, C'\rangle\, ds$$

   prove that $C'(L) \in (T_{c(L)}A)^{\perp}$, clearly state any result you use [6]

6. Let $f : (M, g) \to I\!R$ be a smooth function on a Riemannian manifold. The Hessian of f, at P is defined as follows: Let $X, y \in T_P M$ and let $\tilde{X}, \tilde{Y}$ be vector fields extending X, y. Then

$$D^2 f|_p(X, y) := (\tilde{X}_P(\tilde{y}(t)) - (\nabla_{\tilde{X}} \tilde{y}, (f)$$

a) Prove that $D^2 f$ is a tensor. i.e. $D^2 f(X, y)$ doesn't depend on the extensions $\tilde{X}$ and $\tilde{y}$.
b) Prove that $D^2 f$ is symmetric.
c) If P is a local minimum of f, prove that
$D^2 f|_p(X, X) \geq 0 \qquad \forall X \in T_P M$.

Hint: Consider a geodesic σ with $\sigma(0) = P, \quad \sigma'(0) = X$. [11]